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Orthogonal almost-complex structures of minimal energy

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A nuestro querido Domingo Toledo en su cumpleaños 60

Abstract In this article we apply a Bochner type formula to show that on a compact conformally flat riemannian manifold (or half-conformally flat in dimension 4) certain types of orthogonal almost-complex structures, if they exist, give the absolute minimum for the energy functional. We give a few examples when such minimizers exist, and in particular, we prove that the standard almost-complex structure on the round S^6 gives the absolute minimum for the energy. We also discuss the uniqueness of this minimum and the extension of these results to other orthogonal G-structures.

Keywords Orthogonal almost-complex structure · Conformally flat · Anti-self-dual metric · Nearly-Kahler

Mathematical Subject Classification 53C10 · 53C15 · 53C55

1 Introduction

Let (M^{2n}, g) be a Riemannian manifold. An orthogonal almost-complex structure on M is an automorphism of the tangent bundle $J\colon TM \to TM$ which is orthogonal with respect to g and satisfies $J^2 = -id_{TM}$. The combination (g, J) is also called an "almost-hermitian structure" on M.

Associated with such a structure is the Kähler form $\omega = g(J\cdot, \cdot)$ (or "J with its indices lowered by g") and the energy $E(\omega)$, defined for a compact manifold by

$$E(\omega) = \int_M \|\nabla\omega\|^2 vol,$$

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where $\nabla\omega$ is the covariant derivative of ω with respect to the Levi-Civita connection associated with g, and *vol* is the volume element associated with g.[1]

A natural problem to consider in this context is that of the critical points of the energy functional, for a fixed (M, g). In particular, one seeks orthogonal almost-complex structures J of minimal energy.

For $n = 1$, i.e., (M, g) an oriented riemannian surface, J is unique (up to a sign) and $E(\omega) = 0$. For $n > 1$, $E(\omega) \geq 0$ with equality if and only if $\nabla\omega \equiv 0$, which is the Kähler condition, i.e., J is integrable and ω is closed.

If (M, g) does not admit a Kähler metric then we do not know in general if a minimum occurs, let alone its value. However, in case (M, g) is *conformally flat* (or "half-conformally-flat" for $\dim M = 4$), we are able to derive a useful sufficient condition for the existence of an energy minimizing J and a formula for its energy in terms of the total scalar curvature of g. This is the main result of this paper (Theorem 1).

The key ingredient for the proof is a general Bochner-type formula for orthogonal G-structures previously published in [12] and [3]. Briefly, we consider the Gray-Hervella decomposition of $\nabla\omega$, i.e., its decomposition into the direct sum of four U_n-irreducible components

$$\nabla\omega = \sum_{i=1}^{4} (\nabla\omega)_i,$$

and the corresponding

$$E(\omega) = \sum_{i=1}^{4} E_i(\omega),$$

where

$$E_i(\omega) = \int_M \|(\nabla\omega)_i\|^2 vol, \quad i = 1, \ldots, 4.$$

The Bochner-type formula of [3] implies, under the stated condition of conformal flatness on (M, g), that

$$2E_1(\omega) - E_2(\omega) + (n-1)E_4(\omega) = const., \tag{1}$$

where "*const.*" is some positive multiple of the total scalar curvature of (M, g).

It follows immediately from Formula (1) that the vanishing of certain components of $\nabla\omega$ implies that J is an energy minimizer. In particular, it follows from formula (1) that *the standard orthogonal almost-complex-structure on the 6-sphere S^6, equipped with its standard (round) metric, is an energy minimizer* (see Theorem 2). Incidentally, this result contradicts that of [19], where it is claimed that this structure is not even a local minimizer.

We notice that our formula (1) and its implications is quite similar to other variational problems of geometric origin, such as the Yang-Mills equations in dimension 4 and harmonic maps between Kähler manifolds. In these problems, as in ours, there is a natural decomposition of the "energy" into several components, and one can show that a certain linear combination of these components is identically constant. It follows

[1] Thinking of J as a section of the twistor fibration over M endowed with its natural riemannian metric, this definition is equivalent to $E'(J) = \int_M \|dJ\|^2$; i.e., $E' = aE + b$ where a, b are a pair of constants depending only on the dimension of M.

that structures for which certain components of the energy vanish are absolute minima. In this way one sees that self-dual and anti-self-dual connections on 4-manifolds (the instantons) form the minima of the Yang-Mills functional, and holomorphic or anti-holomorphic maps between Kähler manifolds are the minima of the harmonic map energy (in their homotopy class). Furthermore, in these cases, as in ours, the condition for being a minimum is a first order differential equation on the structure in question, whereas the Euler Lagrange equations for general critical points of the energy are second-order PDE.

In the rest of this article, we first explain how to arrive at Eq. (1) and the precise conditions under which it applies, and then use formula (1) to give several examples of orthogonal almost-complex structures that realize the absolute minimum of the energy.

In the last section of the article we explain how to extend our results to similar problems of "G-structures with minimal energy", such as G_2 and $Spin_7$ structures.

2 A Bochner formula

Let (M^{2n}, g, J) be an almost-hermitian manifold and $\omega = g(J\cdot, \cdot)$ its associated Kähler form. We give here a brief review of the results of [3] concerning such structure.

We use the abbreviated notation Λ^k for the bundle of real k forms on M, $\Lambda^{p,q}$ for the bundle of complex forms of type (p, q) and $[[\Lambda^{p,q}]]$ for the real forms in $\Lambda^{p,q}$.

In this notation, ω is a section of $[[\Lambda^{1,1}]]$, and $\nabla\omega$ is a section of the bundle $\mathcal{W} := \Lambda^1 \otimes [[\Lambda^{2,0}]]$. This bundle decomposes into four subbundles

$$\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_4,$$

the so-called Gray–Hervella decomposition [11], corresponding to the decomposition into irreducibles of the U_n-representation which gives rise to $\mathcal{W}$. We have

- $\mathcal{W}_1 = [[\Lambda^{3,0}]]$;
- $\mathcal{W}_2$ = the real part of the image of $\left(\Lambda^{1,0}\right)^{\otimes 3}$ under the Young symmetrizer $(1 - (23))(1 + (12))$;
- $\mathcal{W}_3$ = real part of the "primitive" part of $\Lambda^{1,2}$ (kernel of the contraction in the first and second entries);
- $\mathcal{W}_4 \cong \Lambda^1$, given by the image of Λ^1, inside of $\mathcal{W}$, of the adjoint of the contraction $\mathcal{W} \subset \Lambda^1 \otimes \Lambda^2 \to \Lambda^1$.

Note . For $n = 1$, $\mathcal{W} = 0$; for $n = 2$, $\mathcal{W}_1 = \mathcal{W}_3 = 0$.

Corresponding to the decomposition of $\mathcal{W}$ is the decomposition of $\nabla\omega$,

$$\nabla\omega = (\nabla\omega)_1 + \cdots + (\nabla\omega)_4,$$

i.e., $(\nabla\omega)_i$ is a section of $\mathcal{W}_i$, $i = 1, \ldots, 4$. It is important to notice that the irreducible U_n-modules giving rise to $\mathcal{W}_i$ are pairwise non-isomorphic, hence the $(\nabla\omega)_i$ are pairwise orthogonal.

The components $(\nabla\omega)_i$ carry important geometric information about the almost-complex structure; for example, J is integrable iff $(\nabla\omega)_1 = (\nabla\omega)_2 = 0$, i.e., $\nabla\omega \in \mathcal{W}_3 \oplus \mathcal{W}_4$ (or "J is of type $\mathcal{W}_3 \oplus \mathcal{W}_4$"). The structure is symplectic ($d\omega = 0$), or "almost Kähler", iff $\nabla\omega \in \mathcal{W}_2$ and is nearly Kähler if $\nabla\omega \in \mathcal{W}_1$, i.e., $\nabla\omega = d\omega$.

Now, when M is compact, corresponding to the decomposition of $\nabla\omega$ is the decomposition of the energy,

$$E(\omega) = E_1(\omega) + E_2(\omega) + E_3(\omega) + E_4(\omega),$$

where

$$E_i(\omega) := \int_M \|(\nabla\omega)_i\|^2.$$

In [12] and [3], via two different arguments, the following formula for an arbitrary (M^{2n}, g, J) was obtained:

$$2E_1(\omega) - E_2(\omega) + (n-1)E_4(\omega) = \frac{1}{2}\int_M \mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp), \tag{2}$$

where $\mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp)$ means the trace of the $(\mathfrak{u}_n^\perp, \mathfrak{u}_n^\perp)$-block of the curvature operator $\mathcal{R}$: $\Lambda^2 = \mathfrak{u}_n \oplus \mathfrak{u}_n^\perp \to \mathfrak{u}_n \oplus \mathfrak{u}_n^\perp$.

Remarks

- For $n = 2$, since only the E_2 and E_4 components exist, i.e., $E(\omega) = E_2(\omega) + E_4(\omega)$, the formula reduces to

$$-E_2(\omega) + E_4(\omega) = \frac{1}{2}\int_M \mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp).$$

- As pointed out by the referee, putting together formulae (12) and (23) of [10], one gets the following pointwise formula:

$$\frac{2}{3}\|(d\omega)^{(3,0)}\|^2 - 4\|N_0\|^2 + \|\theta\|^2 + 2\delta\theta = \frac{2(n-1)}{2n-1}s - 2\langle W(\omega), \omega\rangle,$$

  where
  - $d\omega^{(3,0)}$ is the component of $d\omega$ inside $\mathcal{W}_1 = [[\Lambda^{3,0}]]$; $d\omega^{(3,0)} = (\nabla\omega)_1$.
  - N_0 is the component of the Nijenhuis tensor in $\mathcal{W}_2$. That is, the Nijenhuis tensor N can be identified with the projection of $\nabla\omega$ on $\mathcal{W}_1 \oplus \mathcal{W}_2$, and N_0 with $(\nabla\omega)_2$.
  - $\theta = J\delta\omega \in \Lambda^1$ is the Lee form, and can be identified with $(\nabla\omega)_4$. In fact, in [3], it is computed that $\|\theta\|^2 = \|\delta\omega\|^2 = 2(n-1)\|(\nabla\omega)_4\|^2$.
  - One can verify as in [6] that

$$\langle W(\omega), \omega\rangle = \frac{1}{2(2n-1)}[(2n-1)s^* - s]$$

    and thus, that the right hand side of the above pointwise formula equals $4\mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp)$.

  Integrating the above formula, one obtains (2). Notice that, from this description, Lemma 1 below follows immediately.

In general, the decomposition $\Lambda^2 = \mathfrak{u}_n \oplus \mathfrak{u}_n^\perp$ depends on J, hence the same dependence occurs for the right-hand side of the above formula. Nevertheless, when (M, g) is conformally flat (or half conformally flat in dimension 4), we have the following:

Lemma 1 *Let (M^{2n}, g) be a riemannian manifold with Weyl tensor W and an orthogonal almost-complex structure J. If*

- *$n \geq 3$ and $W = 0$ (i.e. (M, g) is conformally flat), or*
- *$n = 2$ and $W^+ = 0$ (i.e. (M, g) is half-conformally flat, or anti-self-dual, using the orientation induced by J),*

then

$$tr(\mathcal{R}, \mathfrak{u}_n^\perp) = \frac{2n-2}{2n-1} s,$$

where s is the scalar curvature.

Proof This is well known (see for example, [10] or [6]), so we give here only a sketch: by definition, $\mathrm{tr}(\mathcal{R}, \mathfrak{u}_n^\perp)$ is a U_n-invariant functional on the space of curvature type tensors. Representation theory tells us that there are two linearly independent such invariants, but restricted to the space of curvature type tensors with vanishing Weyl tensor W (or vanishing W^+ in dimension 4) there is a unique U_n invariant (up to a constant). The exact value of the constant may be evaluated by computing it on any example (we used the real hyperbolic $2n$-space). □

Corollary 1 *Let (M^{2n}, g) be a compact Riemannian manifold such that*

- *$n \geq 3$ and (M, g) is conformally flat, or*
- *$n = 2$ and (M, g) is anti-self-dual.*

Then every almost-complex structure J, orthogonal with respect to g, satisfies

$$2E_1(\omega) - E_2(\omega) + (n-1)E_4(\omega) = C_g,$$

where C_g is a constant depending only on the metric g; in fact, $C_g = \frac{n-1}{2n-1} \int_M s$, where s is the scalar curvature of g.

Note . For $n = 2$, since $E_1 = 0$, the above formula reduces to

$$-E_2(\omega) + E_4(\omega) = C_g.$$

We now state our main result:

Theorem 1 *Let (M^{2n}, g) be a compact Riemannian manifold such that*

- *$n \geq 3$ and (M, g) is conformally flat, or*
- *$n = 2$ and (M, g) is anti-self-dual.*

Then an orthogonal almost-complex structure J_0 on M is an energy minimizer in each of the following three cases:

1. *$n = 3$ and J_0 is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$.*
2. *n is arbitrary and J_0 is of type $\mathcal{W}_4$.*
3. *n is arbitrary and J_0 is of type $\mathcal{W}_2$.*

Furthermore,

- *$E(J_0) = \frac{1}{n-1} C_g$ in each of the first two cases, $E(J_0) = -C_g$ for the third;*
- *if one of the above types of minimizers exists on (M, g), then any other minimizer is necessarily of the same type.*

Remarks

1. For $n = 2$, type $\mathcal{W}_4$ means J_0 is integrable.
2. For all $n \geq 2$, type $\mathcal{W}_2$ means the associated Kähler form ω_0 is closed, i.e., (M, ω_0) is symplectic.
3. Note that $\mathcal{W}_1$ is also of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, hence for $n = 3$, a J_0 of type $\mathcal{W}_1$ is a minimizer. But the theorem does not exclude in this case the existence of another minimizer of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ which is not of type $\mathcal{W}_1$ (see the example of S^6 below).
4. A conformal change of the metric only affects the $\mathcal{W}_4$ component of $\nabla\omega$ (see for example [11]). Hence any minimizing J_0 of the first two types in the theorem is an energy minimizer with respect to all metrics in the conformal class of g. See more about this in Sect. 3.5 below.
5. There do exist riemannian manifolds that do not admit almost-complex structures of the indicated types, thus for them it is not clear if minimizers exist or not (see the example below of hyperbolic 4-manifolds).

Proof Let J be an arbitrary almost-complex structure orthogonal with respect to g and let ω, ω_0 be the Kähler forms of J, J_0 (resp.).

1. If $n = 3$ and J_0 is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, then

$$\begin{aligned} E(\omega) &= E_1(\omega) + E_2(\omega) + E_3(\omega) + E_4(\omega) \\ &\geq E_1(\omega) - \tfrac{1}{2}E_2(\omega) + E_4(\omega) \\ &= C_g/2 = E_1(\omega_0) - \tfrac{1}{2}E_2(\omega_0) + E_4(\omega_0) \\ &= E_1(\omega_0) + E_4(\omega_0) = E(\omega_0). \end{aligned}$$

2. If n is arbitrary and J_0 is of type $\mathcal{W}_4$, then

$$\begin{aligned} E(\omega) &= E_1(\omega) + E_2(\omega) + E_3(\omega) + E_4(\omega) \\ &\geq \tfrac{2}{n-1}E_1(\omega) - \tfrac{1}{n-1}E_2(\omega) + E_4(\omega) \\ &= \tfrac{C_g}{n-1} = \tfrac{2}{n-1}E_1(\omega_0) - \tfrac{1}{n-1}E_2(\omega_0) + E_4(\omega_0) \\ &= E_4(\omega_0) = E(\omega_0). \end{aligned}$$

3. Finally, if J_0 is of type $\mathcal{W}_2$ (symplectic) then

$$\begin{aligned} E(\omega) &= E_1(\omega) + E_2(\omega) + E_3(\omega) + E_4(\omega) \\ &\geq -2E_1(\omega) + E_2(\omega) - (n-1)E_4(\omega) \\ &= -C_g = -2E_1(\omega_0) + E_2(\omega_0) - (n-1)E_4(\omega_0) \\ &= E_2(\omega_0) = E(\omega_0). \end{aligned}$$

One can also read out easily from these calculations the exact value of the minimal energy. For example, for $n = 3$ and a J_0 of type $\mathcal{W}_1 \oplus \mathcal{W}_4$,

$$E(\omega_0) = E_1(\omega_0) + E_4(\omega_0) = E_1(\omega_0) - \frac{1}{2}E_2(\omega_0) + E_4(\omega_0) = C_g/2.$$

The other cases are handled similarly.

For the last statement of the theorem, notice (for example) that if $n = 3$ and a J_0 of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ exists, then if J is another minimizer we will have equality in the first inequality above; thus, $E_2(\omega) = E_3(\omega) = 0$ and J is of type $\mathcal{W}_1 \oplus \mathcal{W}_4$. The other cases are handled similarly. □

3 Examples

Here we go through the various classes of minimal energy almost-hermitian structures indicated in Theorem 1 and try to find non-Kähler examples in each case.

3.1 S^6 ($n = 3$, type $\mathcal{W}_1$)

The round 6-sphere has a natural compatible almost-complex structure, J_C, given by Cayley cross-product in $\mathbb{R}^7$ (thought of as imaginary Cayley numbers): at a point $u \in S^6$ and $v \in T_uS^6$, $J_C(v) = u \times v$. It is well known that such structure is nearly Kähler (i.e., of type $\mathcal{W}_1$), and so, according to the previous theorem, *realizes the absolute minimum of the energy among all almost-complex structures orthogonal with respect to the round metric.* Following Remark 4 of the last section we can say a little more.

Definition 1 *Let (M^{2n}, g) be a riemannian manifold. We say that two almost complex structures J, J_0 on M are conformally equivalent if there exists a conformal diffeomorphism $\phi\colon M \to M$ ($\phi^* g = e^\lambda g$) such that $J_0 = d\phi \circ J \circ d\phi^{-1}$.*

Then we have:

Theorem 2 *Any almost complex structure on S^6 which is conformally equivalent to the Cayley almost complex structure is energy minimizing with respect to all metrics in the conformal class of the round metric.*

Remarks In [5] it has been shown that J_C is also a *volume* minimizer, among all sections of the twistor fibration.

Concerning the uniqueness of this minimizer we have the following theorem of Friedrich [7]:

Theorem 3 *J_C is the unique nearly-Kähler structure on the round S^6, up to an isometry; i.e., for any almost complex structure J on S^6 which is nearly Kähler wrt the round metric there exists an isometry $\phi \in SO_7$ such that $J = d\phi \circ J_C \circ d\phi^{-1}$.*

Hence J_C is the unique (up to an isometry) energy minimizer on the round S^6 of type $\mathcal{W}_1$. We know from Theorem 1 that any other minimizer should be of type $\mathcal{W}_1 \oplus \mathcal{W}_4$, but we do not know if there is actually any one which is not conformally equivalent to J_C.

3.2 Hermitian manifolds (type $\mathcal{W}_4$)

3.2.1 Conformal deformations of Kahler manifolds

One way to obtain a non-Kähler hermitian manifold (in fact, of type $\mathcal{W}_4$) is to start with a Kahler manifold and deform its metric conformally (by a non-constant factor).

This adds a $\mathcal{W}_4$ component to $\nabla\omega$ (see Sect. 3.5 below), hence, if one starts with a conformaly flat Kähler manifold (or ASD manifold for $n = 2$), we obtain in this way a non-Kähler energy minimizer.

Now it turns out that for $n \geq 3$, a conformaly flat compact Kähler manifold is in fact flat [2, 2.68], so these are all the examples of non-Kähler energy minizers we can obtain by this method for $n \geq 3$.

For $n = 2$, we would like to start with an ASD Kähler manifold and again deform the metric conformally to obtain an ASD hermitian (non-Kahler) manifold. Now for a (complex) two-dimensional Kahler manifold the ASD condition is equivalent to the vanishing of the scalar curvature, and there are many examples of such SFK ("scalar flat Kahler") manifolds (see for example [15]).

Interestingly, in case the first Betti number of M is even, a result of Boyer [4] states that this method is the *only* way to obtain an ASD hermitian manifold:

if a compact ASD Hermitian manifold has even first Betti number then there is a conformal change of the metric that transforms it into a Kähler metric of zero scalar curvature.

For a manifold with odd first Betti number this method clearly cannot work (since such a manifold cannot admit a Kähler metric), but we have the following examples of LeBrun [14]:

There exist ASD Hermitian metrics on the k-fold blow-ups

$$(S^1 \times S^3)\#\overline{\mathbb{C}P^2}\# \cdots \#\overline{\mathbb{C}P^2}.$$

3.2.2 Hopf manifolds

The product metric on $S^{2n-1} \times S^1$ is conformally flat. This manifold does not admit a Kähler metric (the first Betti number is odd), but does admit many orthogonal complex structures coming from actions of $\mathbb{Z}$ on $\mathbb{C}^n \setminus \{0\}$ by conformal linear maps. Hence all of these structures are locally conformally Kähler, i.e., of type $\mathcal{W}_4$ (see [18]) and thus of minimal energy.

3.3 Symplectic manifolds (type $\mathcal{W}_2$)

For $n \geq 3$, we do not know of any examples of conformally flat, symplectic, non-Kähler compact manifolds.

For $n = 2$, we recall Armstrong's deformation argument [1] to produce examples of ASD non-Kähler symplectic structures (we thank V. Apostolov for explaining this to us):

Start with a scalar flat Kähler (SFK) metric (g_0, J_0) on a four manifold. Such a metric can be shown to exist on certain complex surfaces; for example, on a blow-up of a generic ruled complex surface, see [15]. Such a metric is ASD, since, as we have recalled before, for a Kähler metric ASD is equivalent to scalar flat.

The idea is to show that on such a manifold there are ASD deformations of the conformal class $[g_0]$ admitting symplectic structures which are not Kähler. To this end we consider two types of deformations:

(a) SFK deformations of (g_0, J_0).
(b) ASD deformations of the conformal class $[g_0]$.

These moduli spaces have been studied by [15] and [13]. Just by comparing their dimensions we see that (b) is larger than (a) on our manifold. So there are ASD classes $[g]$ arbitrarily close to $[g_0]$ which are not Kähler. Let us see that such a $[g]$ admits a compatible symplectic structure. Note first, that on a four-manifold, the condition on a symplectic form ω to be "admitted" by a conformal class $[g]$ is simply that ω be self-dual wrt $[g]$ (because this implies that ω is the Kähler form associated with some orthogonal almost complex structure and some metric in the conformal class of g). Now let η be the harmonic representative, wrt g, of the deRham cohomology class of ω_0, and let ω be the SD part of η. Then ω is still harmonic, thus closed, and SD wrt g. If g is near g_0 then ω is near ω_0 and is non-degenerate, hence symplectic.

3.4 Conformal change of the metric

Consider an almost-hermitian manifold (M, g, J), and a conformal change of the metric: thus, set $g' = \lambda g$ with $\lambda\colon M \to \mathbb{R}^+$. Note that J is orthogonal wrt g' as well.

Denoting by ω' and ∇' the Kähler form and Levi–Civita connection of the almost-hermitian manifold (M, g', J), we have the following relation (see [9] and [11])

$$\nabla'\omega' = \lambda\nabla\omega + \epsilon(\lambda)$$

where $\epsilon(\lambda)$ is a certain tensor in $\mathcal{W}_4$ depending on λ and $d\lambda$ and such that $\epsilon(\lambda) \equiv 0$ if and only if λ is constant. Observe that neither $\mathcal{W}$ nor its decomposition $\mathcal{W} = \bigoplus \mathcal{W}_i$ change under a conformal change of the metric.

We see that there is no immediate relation that can be deduced between the energies of J with respect to g and g'; in particular, there's no obvious reason why a J which has minimal energy for g should still be a minimizer for g' – even if (g, J) is a Kähler structure on M.

This motivates the following question: *is there a compact manifold M and a conformal class of metrics* $[g]$*, such that M is Kähler for two different (i.e., non-homothetic) metrics in* $[g]$*?*

We do not know the answer to this question in general. However, from Theorem 1 we know that if (M, g, J) is a conformally flat $\mathcal{W}_4$-manifold (or ASDH-manifold in dimension 4, or $\mathcal{W}_1 \oplus \mathcal{W}_4$ conformally flat manifold in dimension 6), then (M, g', J) is still conformally flat (or ASD) and J is of the same type as for g. Thus J will still be of minimum energy for the new metric g'. This leads to the following well known result.

Corollary 2 *Let* $[g]$ *be the conformal class of a metric on a compact manifold. Assume that* $[g]$ *is ASD in dimension 4 or conformally flat in higher dimensions. Then inside of* $[g]$ *there is at most one Kähler metric (up to constant multiples).*

Proof Suppose (g, J) is Kähler. Then, J has minimal energy (namely 0) wrt g. Let $g' = \lambda g$, for some non-constant $\lambda\colon M \to \mathbb{R}^+$. Then (g', J) is of type $\mathcal{W}_4$ and hence, by Theorem 1, J is an energy minimizer wrt g'. Since λ is not constant then $\epsilon(\lambda)$ is not zero and so (g', J) has positive energy. It follows that g' cannot be Kähler, since this would imply the existence of a J' with zero energy wrt to g'. □

3.5 Hyperbolic 4-manifolds

Let (M^4, g) be a compact real hyperbolic 4-manifold. It is known that many of these admit orthogonal almost complex structures (iff its Euler characteristic is divisible by four), but cannot admit neither a compatible complex structure [3,10] nor a compatible symplectic structure [17]. We do not know whether these manifolds have a minimizer or not.

4 G_2 and $Spin_7$ structures

A G_2-structure on a compact 7-manifold M is given by a certain 3-form ϕ. All G_2-structures on M compatible with a fixed metric g and orientation are given by sections of a fibre bundle with fiber $SO_7/G_2 \simeq \mathbb{R}P^7$. We can, again, define the energy of a G_2-structure as the L^2-norm of $\nabla\phi$.

In this case the formula is (see [3]),

$$6E_1 + 5E_7 - E_{14} - E_{27} = \frac{2}{3}\int_M s.$$

Thus any nearly-parallel (i.e., type $\mathcal{W}_1$) G_2-structure on M^7 minimizes energy among all G_2-structures sharing the same metric. For example, the standard G_2-structure on S^7 is of type $\mathcal{W}_1$ [8], hence is minimal among all G_2 structures compatible with the round metric.

Similarly, if M^7 admits a $\mathcal{W}_{14} \oplus \mathcal{W}_{27}$ G_2-structure, that structure will have minimal energy.

For $Spin_7$-structures the formula reads

$$6E_8 - E_{48} = \frac{1}{6}\int_M s.$$

Hence the natural $Spin_7$-structure on $S^7 \times S^1$ has minimal energy since it is of type W_8 [16].

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